Online Appendix

A Framework for Eliciting, Incorporating, and Disciplining Identification Beliefs in Linear Models

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A Proofs

Proof of Lemma 2.1. By the definitions of (u, v, ζ, w) and the properties of covariance,

$$\begin{aligned}
\sigma_{uw} &= [\text{Cov}(u, \widetilde{w}) - \psi \text{Cov}(u, T^*)] \\
\sigma_{\zeta w} &= [\text{Cov}(z, \widetilde{w}) - \psi \text{Cov}(z, T^*)] - [\text{Cov}(\widetilde{w}, \mathbf{x}') - \psi \text{Cov}(T^*, \mathbf{x}')] \, \boldsymbol{\varphi}_z \\
\sigma_{vw} &= [\text{Cov}(T^*, \widetilde{w}) - \psi \text{Var}(T^*)] - \pi \, [\text{Cov}(z, \widetilde{w}) - \psi \text{Cov}(z, T^*)] - [\text{Cov}(\mathbf{x}', \widetilde{w}) - \psi \text{Cov}(\mathbf{x}', T^*)] \, \boldsymbol{\eta}.
\end{aligned}$$

By the definition of ψ, $[\text{Cov}(T^*, \widetilde{w}) - \psi \text{Var}(T^*)] = 0$. Moreover, by Assumption 2.2 all of the remaining terms in square brackets likewise equal zero. Thus, $\sigma_{uw} = \sigma_{\zeta w} = \sigma_{vw} = 0$. Next, $\sigma_{v\zeta} = \text{Cov}(v, z) - \text{Cov}(v, \mathbf{x}')\boldsymbol{\varphi}_z = 0$ because $\text{Cov}(v, z)$ and $\text{Cov}(v, \mathbf{x}') = 0$ by Assumption 2.1 (ii). Finally, $\text{Cov}(\mathbf{x}, w) = [\text{Cov}(\mathbf{x}, \widetilde{w}) - \psi \text{Cov}(\mathbf{x}, T^*)] = \mathbf{0}$ by Assumption 2.2 and the definition of w. □

Proof of Lemma 2.2. Substituting (2) and the reduced form for z into (1),

$$y = \mathbf{x}'\boldsymbol{\varphi}_y + \varepsilon = \mathbf{x}' \left[\beta \left(\pi \boldsymbol{\varphi}_z + \boldsymbol{\eta}\right) + \boldsymbol{\gamma}\right] + \left[\beta(\pi\zeta + v) + u\right]$$

by equating with the reduced form equation for y from (4). Similarly, substituting (2) and the reduced form for z into (5) gives

$$T = \mathbf{x}'\boldsymbol{\varphi}_T + \xi = \mathbf{x}' \left[\tau \mathbf{e}_1 + (1+\psi)(\pi \boldsymbol{\varphi}_z + \boldsymbol{\eta})\right] + \left[(1+\psi)(\pi\zeta + v) + w\right]$$

by equating with the reduced form equation for T from (4). Now, $\mathbb{E}(w) = 0$ by construction, and since $\mathbf{x}$ includes a constant, ζ and v are likewise mean zero. The result follows since (ζ, v, u) are uncorrelated with $\mathbf{x}$ by Assumption 2.1 and Lemma 2.1. □

Proof of Lemma 2.3. The result follows immediately by inspection of (9)–(13) and the equality

$$\begin{bmatrix} \sigma_u^2 & \sigma'_{uv} & \sigma_{u\zeta} \\ \sigma'_{uv} & (\sigma'_v)^2 & 0 \\ \sigma_{u\zeta} & 0 & \sigma_\zeta^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \left(\frac{1+\psi}{1+\psi'}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_u^2 & \sigma_{uv} & \sigma_{u\zeta} \\ \sigma_{uv} & \sigma_v^2 & 0 \\ \sigma_{u\zeta} & 0 & \sigma_\zeta^2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \left(\frac{1+\psi}{1+\psi'}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

with $(1+\psi) > 0$ and $(1+\psi') > 0$. □

Proof of Proposition 2.1. Substituting (18) into (19) and rearranging, $\widetilde{\beta} = (s_{13} - \sigma_{u\zeta})/s_{23}$, while solving (21) for $\widetilde{\beta}$ gives $\widetilde{\beta} = (s_{12} - \widetilde{\sigma}_{u\xi^*})/\widetilde{\kappa} s_{22}$. Equating these two expressions,

$$\frac{s_{13} - \sigma_{u\zeta}}{s_{23}} = \frac{s_{12} - \widetilde{\sigma}_{u\xi^*}}{\widetilde{\kappa} s_{22}}. \tag{A.1}$$

Similarly, substituting $\widetilde{\beta} = (s_{13} - \sigma_{u\zeta})/s_{23}$ and $\widetilde{\beta}\widetilde{\kappa} s_{22} = (s_{12} - \widetilde{\sigma}_{u\xi^*})$ into (22),

$$(\sigma_u^2 - s_{11}) + \left(\frac{s_{13} - \sigma_{u\zeta}}{s_{23}}\right)(\widetilde{\sigma}_{u\xi^*} + s_{12}) = 0. \tag{A.2}$$

Re-arranging (24) gives $\widetilde{\sigma}_{u\xi^*} = \rho_{u\xi^*}\sigma_u(\widetilde{\kappa} s_{22})^{1/2}$. Substituting this and $\sigma_{u\zeta} = \sigma_u \rho_{u\zeta} s_{33}$ into (A.1)–(A.2),

$$\frac{s_{13} - \sigma_u \rho_{u\zeta} s_{33}}{s_{23}} = \frac{s_{12} - \rho_{u\xi^*}\sigma_u(\widetilde{\kappa} s_{22})^{1/2}}{\widetilde{\kappa} s_{22}} \tag{A.3}$$

$$(\sigma_u^2 - s_{11}) + \left(\frac{s_{13} - \sigma_u \rho_{u\zeta} s_{33}}{s_{23}}\right)\left[\rho_{u\xi^*}\sigma_u(\widetilde{\kappa} s_{22})^{1/2} + s_{12}\right] = 0. \tag{A.4}$$

Substituting (A.3) into (A.4) and re-arranging, we obtain

$$\sigma_u^2 = \frac{s_{11}(\widetilde{\kappa} - r_{12}^2)}{\widetilde{\kappa}(1 - \rho_{u\xi^*}^2)}. \tag{A.5}$$

The result follows by substituting the positive square root of (A.5) into (A.3) and solving the resulting expression for $\rho_{u\zeta}$. □

Lemma A.1. *Under Assumptions 2.1–2.3,*

(a) $\widetilde{\sigma}_v^2 = s_{22}(\widetilde{\kappa} - r_{23}^2)$

(b) $\rho_{uv} = \widetilde{\rho}_{uv} = \dfrac{\rho_{u\xi^*}\sqrt{\widetilde{\kappa}} - \rho_{u\zeta} r_{23}}{\sqrt{\widetilde{\kappa} - r_{23}^2}}$

where r_{23} is as defined in Proposition 2.1, $\rho_{uv} \equiv Cor(u, v)$, and $\widetilde{\rho}_{uv} \equiv Cor\big(u, (1+\psi)v\big)$.

Proof of Lemma A.1(a). By (18), $r_{23}^2 \equiv \mathrm{Cor}(\xi, \zeta)^2 = \widetilde{\pi}^2 s_{33}/s_{22}$. By (11) and (17), $s_{22}\widetilde{\kappa} = \widetilde{\pi}^2 s_{33} + \widetilde{\sigma}_v^2$. The result follows by combining these and re-arranging. □

Proof of Lemma A.1(b). By (15) and (24),

$$\rho_{u\xi^*} = \left(\frac{\widetilde{\sigma}_v}{\sqrt{\widetilde{\kappa} s_{22}}}\right)\rho_{uv} + \left(\frac{\widetilde{\pi}\sigma_\zeta}{\sqrt{\widetilde{\kappa} s_{22}}}\right)\rho_{u\zeta}. \tag{A.6}$$

By manipulating Lemma A.1(a), we obtain $\widetilde{\sigma}_v/\sqrt{\widetilde{\kappa} s_{22}} = \sqrt{1 - r_{23}^2/\widetilde{\kappa}}$. From the proof of Lemma A.1(a), $r_{23}^2 = \widetilde{\pi}^2 s_{33}/s_{22}$, so that $\widetilde{\pi}\sigma_\zeta/\sqrt{\widetilde{\kappa} s_{22}} = r_{23}/\sqrt{\widetilde{\kappa}}$. The result follows by substituting these two equalities into (A.6) and solving for ρ_{uv}. Because $\sigma_v^2 > 0$ if and only if $\widetilde{\sigma}_v^2 > 0$, and $\widetilde{\sigma}_v^2 > 0$ if and only if $\widetilde{\kappa} > r_{23}^2$ by Lemma A.1(a), the quantity under the radical is always strictly positive making division by $\sqrt{\widetilde{\kappa} - r_{23}^2}$ permissible here. □

Lemma A.2. *Under Assumptions 2.1, 2.2, and 2.3(i), the matrix Ω_{11} defined in Lemma 2.1 is positive definite if and only if $\sigma_u^2, \sigma_v^2, \sigma_\zeta^2 > 0$ and $\rho_{uv}^2 + \rho_{u\zeta}^2 < 1$.*

Proof of Lemma A.2. By Lemma 2.1, Ω_{11} is positive definite if and only if

$$\sigma_u^2 > 0 \tag{A.7}$$
$$\sigma_u^2\sigma_v^2 - \sigma_{uv}^2 > 0 \tag{A.8}$$
$$\sigma_\zeta^2(\sigma_u^2\sigma_v^2 - \sigma_{uv}^2) - \sigma_v^2\sigma_{u\zeta}^2 > 0. \tag{A.9}$$

For the "if" direction, first note that by (A.7) we can rearrange (A.8) to yield $\sigma_v^2 > \sigma_{uv}^2/\sigma_u^2 \geq 0$. Dividing through by σ_v^2, this implies that $|\rho_{uv}| < 1$. Now, since both σ_u^2 and σ_v^2 are strictly positive, we can divide both sides of (A.9) through by $\sigma_v^2\sigma_u^2$ to obtain $\sigma_\zeta^2(1 - \rho_{uv}^2) > \sigma_{u\zeta}^2/\sigma_u^2 \geq 0$. Since $\rho_{uv}^2 < 1$, this implies $\sigma_\zeta^2 > 0$. Thus, dividing (A.9) through by $\sigma_v^2\sigma_u^2\sigma_\zeta^2$ and rearranging we find that $\rho_{uv}^2 + \rho_{u\zeta}^2 < 1$. For the "only if" direction, $\rho_{uv}^2 + \rho_{u\zeta}^2 < 1$ implies $\rho_{uv}^2 < 1$. Mutiplying both sides by $\sigma_u^2\sigma_v^2$ gives $\sigma_u^2\sigma_v^2\rho_{uv} < \sigma_u^2\sigma_v^2$ since $\sigma_u^2, \sigma_v^2 > 0$. Substituting $\rho_{uv}^2 = \sigma_{uv}^2/(\sigma_u^2\sigma_v^2)$ and rearranging implies (A.8). Equation A.9 follows similarly, by multiplying both sides of $\rho_{uv}^2 + \rho_{u\zeta}^2 < 1$ by $\sigma_u^2\sigma_v^2\sigma_\zeta^2$ and rearranging. □

Proof of Proposition 2.2. By Assumption 2.3 (ii), Ω_{11} is positive definite. Thus, by Lemma A.2 $\sigma_v^2, \sigma_u^2, \sigma_\zeta^2 > 0$ and $\rho_{uv}^2 + \rho_{u\zeta}^2 < 1$. Since $\sigma_v^2 > 0$ and $\psi \neq -1$ by Assumption 2.1 (v), it follows that $\widetilde{\sigma}_v^2 \equiv (1+\psi)^2\sigma_v^2 > 0$. Hence, by Lemma A.1(a), $\widetilde{\kappa} > \rho_{Tz}^2$. Similarly, since $\sigma_u^2 > 0$, it follows from Equation A.5 in the proof of Proposition 2.1 that $\widetilde{\kappa} > r_{12}^2$. Combining these, we see that $\widetilde{\kappa} > \max\{r_{12}^2, r_{23}^2\}$. By Lemma A.1(a), $\rho_{uv}^2 + \rho_{u\zeta}^2 < 1$ is equivalent to

$$\left(\frac{\rho_{u\xi^*}\sqrt{\widetilde{\kappa}} - \rho_{u\zeta}r_{23}}{\sqrt{\widetilde{\kappa} - r_{23}^2}}\right)^2 + \rho_{u\zeta}^2 < 1 \tag{A.10}$$

Putting the terms of (A.10) over a common denominator and rearranging,

$$\rho_{u\xi^*}^2 + \rho_{u\zeta}^2 - \frac{2\rho_{u\xi^*}\rho_{u\zeta}r_{23}}{\widetilde{\kappa}^{1/2}} < \frac{\widetilde{\kappa} - \rho_{23}^2}{\widetilde{\kappa}}$$

using the fact that $\widetilde{\kappa} > r_{23}^2$. Completing the square,

$$\left(\rho_{u\zeta} - \frac{\rho_{u\xi^*}r_{23}}{\widetilde{\kappa}^{1/2}}\right)^2 < (1 - \rho_{u\xi^*}^2)\left(\frac{\widetilde{\kappa} - r_{23}^2}{\widetilde{\kappa}}\right).$$

Now, using (2.1) to substitute for $(\rho_{u\zeta} - \rho_{u\xi^*} r_{23}/\sqrt{\widetilde{\kappa}})$, we find that

$$(r_{12}r_{23} - \widetilde{\kappa} r_{13})^2 \left[\frac{1-\rho_{u\xi^*}^2}{\widetilde{\kappa}(\widetilde{\kappa} - r_{12}^2)}\right] < \left(1 - \rho_{u\xi^*}^2\right)\left(\frac{\widetilde{\kappa} - r_{23}^2}{\widetilde{\kappa}}\right)$$

Cancelling a factor of $(1-\rho_{u\xi^*}^2)/\widetilde{\kappa}$ from each side and rearranging

$$(r_{12}r_{23} - \widetilde{\kappa}\rho_{13})^2 - (\widetilde{\kappa} - r_{12}^2)(\widetilde{\kappa} - r_{23}^2) < 0 \tag{A.11}$$

using the fact that $\widetilde{\kappa} > r_{12}^2$. Expanding and simplifying,

$$(r_{13}^2 - 1)\widetilde{\kappa}^2 + (r_{12}^2 + r_{23}^2 - 2r_{12}r_{23}r_{13})\widetilde{\kappa} < 0.$$

Since Σ is positive definite, $r_{13}^2 < 1$. Hence, the preceding inequality defines an interval of values that $\widetilde{\kappa}$ *cannot* take on, an interval bounded by the roots of a quadratic function that opens downwards. To determine these roots, we factorize as follows:

$$\widetilde{\kappa}\left[(r_{13} - 1)\widetilde{\kappa} + \left(r_{12}^2 + r_{23}^2 - 2r_{12}r_{23}r_{13}\right)\right] = 0.$$

Thus one root is zero and the other is L. To complete the proof, we show that $L < 1$ and $L > \max\{r_{12}^2, r_{23}^2\}$. For the first claim, note that the positive definiteness of Σ implies

$$1 - r_{12}^2 - r_{23}^2 - r_{13}^2 + 2r_{12}r_{23}r_{13} > 0.$$

Rearranging this inequality using $r_{13}^2 < 1$ establishes $L < 1$. For the second claim notice that (A.11) is violated at $\widetilde{\kappa} = \max\{r_{12}^2, r_{23}^2\}$. This combined with the fact that the parabola opens downwards establishes that L is greater than both zero and $\max\{r_{12}^2, r_{23}^2\}$. □

Proof of Theorem 2.1. Let $(\rho_{u\zeta}, \rho_{u\xi^*}, \widetilde{\kappa})$ be any triple satisfying $|\rho_{u\xi^*}| < 1$, $\widetilde{\kappa} \in (L, 1]$ and (25). Given this triple, the argument proceeds by constructing errors (u, v, w, ξ^*) and parameter values $(\psi, \tau, \pi, \boldsymbol{\eta}, \boldsymbol{\gamma}, \boldsymbol{\varphi}_T^*, \beta)$ that satisfy Assumptions 2.1–2.3 and generate the observed random variables under (1), (2), and (5). This construction depends on the observable reduced form parameters $(\boldsymbol{\varphi}_y, \boldsymbol{\varphi}_T, \boldsymbol{\varphi}_z)$ and errors $(\varepsilon, \xi, \zeta)$.

The first step constructs w so that $\mathbb{E}(w) = 0$, $\sigma_w^2 = s_{22}(1-\widetilde{\kappa})$, $\text{Cov}(w, \varepsilon) = \text{Cov}(w, \zeta) = 0$, $\text{Cov}(w, \mathbf{x}) = \mathbf{0}$, and $\text{Cov}(w, \xi) = \sigma_w^2$. To this end, let χ be the residual from a projection of ξ on ζ and ε, i.e. $\xi = a\varepsilon + b\zeta + \chi$ with $\text{Cov}(\varepsilon, \chi) = \text{Cov}(\zeta, \chi) = 0$. Next let $\mathscr{W}$ be any random variable with $\mathbb{E}(\mathscr{W}) = 0$ and $\text{Var}(\mathscr{W}) = 1$ that is uncorrelated with χ, ε, ζ, and $\mathbf{x}$. We define w in terms of χ and $\mathscr{W}$ as

$$w = \left(\frac{1-\widetilde{\kappa}}{1-L}\right)\chi + \left[\frac{s_{22}(1-\widetilde{\kappa})(\widetilde{\kappa} - L)}{1-L}\right]^{1/2}\mathscr{W}. \tag{A.12}$$

Note that the constants in (A.12) are both well-defined and non-negative, since $L < \widetilde{\kappa} \leq 1$ by Proposition 2.2. Now, because $\mathbf{x}$ includes a constant, $(\xi, \zeta, \varepsilon)$ are mean zero and hence $\mathbb{E}(w) = 0$ by construction. Moreover, since χ is by construction uncorrelated with ε and ζ, it follows that $\text{Cov}(w, \varepsilon) = \text{Cov}(w, \zeta) = 0$. Similarly, since χ and $\mathscr{W}$ are both uncorrelated with $\mathbf{x}$, so is w. To calculate σ_w^2 and $\text{Cov}(w,\xi)$, note that

$$\text{Var}(\chi) = s_{22} - \left[\begin{array}{cc} s_{12} & s_{23} \end{array}\right] \left[\begin{array}{cc} s_{11} & s_{13} \\ s_{13} & s_{33} \end{array}\right]^{-1} \left[\begin{array}{c} s_{12} \\ s_{23} \end{array}\right] = s_{22}(1-L)$$

from which it follows that

$$\sigma_w^2 = \left(\frac{1-\widetilde{\kappa}}{1-L}\right)^2 s_{22}(1-L) + \left[\frac{s_{22}(1-\widetilde{\kappa})(\widetilde{\kappa}-L)}{1-L}\right] = s_{22}(1-\widetilde{\kappa})$$

and

$$\text{Cov}(w,\xi) = \text{Cov}(w, a\varepsilon + b\zeta + \chi) = \text{Cov}(w,\chi) = \left(\frac{1-\widetilde{\kappa}}{1-L}\right)\text{Var}(\chi) = s_{22}(1-\widetilde{\kappa}) = \sigma_w^2.$$

The second step constructs errors (ξ^*, v, u) and parameters $(\boldsymbol{\varphi}_T^*, \boldsymbol{\eta}, \boldsymbol{\gamma})$ so that (1) generates the observed distribution of y, (2) generates a distribution for T^* that is compatible with our observables, and (5) generates the observed distribution of T. To this end, set

$$\xi^* = \frac{\xi - w}{1+\psi}, \quad v = \frac{\xi - w}{1+\psi} - \pi\zeta, \quad u = \varepsilon - \beta\left(\frac{\xi - w}{1+\psi}\right)$$

and

$$\boldsymbol{\varphi}_T^* = \frac{\boldsymbol{\varphi}_T - \tau\mathbf{e}_1}{1+\psi}, \quad \boldsymbol{\eta} = \frac{\boldsymbol{\varphi}_T - \tau\mathbf{e}_1}{1+\psi} - \pi\boldsymbol{\varphi}_z, \quad \boldsymbol{\gamma} = \boldsymbol{\varphi}_y - \beta\left(\frac{\boldsymbol{\varphi}_T - \tau\mathbf{e}_1}{1+\psi}\right).$$

Substituting the preceding expressions along with the reduced forms for T^* and z and simplifying, we obtain

$$\beta T^* + \mathbf{x}'\boldsymbol{\gamma} + u = \mathbf{x}'\boldsymbol{\varphi}_y + \varepsilon, \quad \pi z + \mathbf{x}'\boldsymbol{\eta} + v = \mathbf{x}'\boldsymbol{\varphi}_T^* + \xi^*, \quad \tau + (1+\psi)T^* + w = \mathbf{x}'\boldsymbol{\varphi}_T + \xi$$

as required. Notice that τ is completely unconstrained in this construction. Moreover, the only restriction imposed on ψ thus far has been $\psi \neq -1$ so that division by $1+\psi$ is well-defined.

The third step sets π and ψ so that our construction satisfies Assumption 2.1. First, we have

$$\text{Cov}(\mathbf{x}, u) = \text{Cov}\left(\mathbf{x}, \varepsilon - \beta\left[\frac{\xi - w}{1+\psi}\right]\right) = \mathbf{0}, \quad \text{Cov}(\mathbf{x}, v) = \text{Cov}\left(\mathbf{x}, \frac{\xi - w}{1+\psi} - \pi\zeta\right) = \mathbf{0}$$

since $\mathbf{x}$ is uncorrelated with the reduced form errors $(\varepsilon, \xi, \zeta)$ by definition, and is likewise uncorrelated with w by construction. This verifies (i) and the first part of (ii). Now set $\pi = s_{23}/[(1+\psi)s_{33}]$. Since $\mathbf{x}$ is uncorrelated with (ζ, ξ, w), it follows that

$$\text{Cov}(z, v) = \text{Cov}\left(\mathbf{x}'\boldsymbol{\varphi}_z + \zeta, \frac{\xi - w}{1+\psi} - \pi\zeta\right) = \frac{s_{23}}{1+\psi} - \pi s_{33} = 0$$

satisfying the second part of (ii). Since $s_{23} \neq 0$, $\pi \neq 0$ satisfying (iii). Since (iv) simply requires that $\mathbf{x}$ include a constant, this requirement is trivially satistied. For (v), since

$T = \tau + (1+\psi)T^* + w$, we have $\text{Cov}(T, T^*) > 0$ for any $\psi > -1$.

The fourth step verifies that our construction satisfies Assumption 2.2. Solving (5) for T^* and combining the result with (3), we obtain $\widetilde{w} = \psi(T + \tau + w)/(1+\psi)$. Accordingly, for any random variable Ξ, we have $\text{Cov}(\Xi, \widetilde{w}) = \psi\text{Cov}(\Xi, T+w)/(1+\psi)$ and $\text{Cov}(\Xi, T^*) = \text{Cov}(\Xi, T-w)/(1+\psi)$. It follows that $\text{Cov}(\Xi, \widetilde{w}) = \psi\text{Cov}(\Xi, T^*)$ if and only if $\text{Cov}(\Xi, w) = 0$. Hence, to verify Assumption 2.2 it suffices to show that $\text{Cov}(u, w) = 0$, $\text{Cov}(z, w) = 0$, and $\text{Cov}(\mathbf{x}, w) = 0$. The first and last of these equalities hold by our construction of w and u above. For the second, we have $\text{Cov}(z, w) = \boldsymbol{\varphi}'_z\text{Cov}(\mathbf{x}, w) + \text{Cov}(\zeta, w) = 0$.

The final step sets $\widetilde{\beta} = (s_{13} - \sigma_{u\zeta})/s_{23}$ to ensure that our construction satisfies Assumption 2.3. By Lemma A.2 it suffices to verify that $\sigma_u^2, \sigma_v^2, \sigma_\zeta^2 > 0$ and $\rho_{uv}^2 + \rho_{u\zeta}^2 < 1$. First, $\sigma_\zeta^2 = s_{33} > 0$ since Σ is positive definite. Next,

$$\begin{aligned}\sigma_v^2 &= \text{Var}\left(\frac{\xi - w}{1+\psi} - \pi\zeta\right) = \left(\frac{1}{1+\psi}\right)^2 \text{Var}(\xi - w) + \pi^2 s_{33} - 2\left(\frac{\pi}{1+\psi}\right)\text{Cov}(\xi - w, \zeta)\\ &= \left(\frac{1}{1+\psi}\right)^2 s_{22}\widetilde{\kappa} + \frac{s_{23}^2}{(1+\psi)^2 s_{33}} - \frac{2s_{23}^2}{(1+\psi)^2 s_{33}} = \left(\frac{1}{1+\psi}\right)^2 s_{22}(\widetilde{\kappa} - r_{23}^2)\end{aligned}$$

by substituting $\pi = s_{23}/\left[(1+\psi)s_{33}\right]$ and using the properties of w from our construction above. Since $L < \widetilde{\kappa} \leq 1$ and $L > r_{23}^2$ by Proposition 2.2, it follows that $\sigma_v^2 > 0$. To establish that $\sigma_u^2 > 0$, we show that our construction satisfies (A.3) and (A.4) from the proof of Proposition 2.1. This implies (A.5) by the argument of Proposition 2.1 and it follows that $\sigma_u^2 > 0$ since $\widetilde{\kappa} > r_{12}^2$. To this end, first note that

$$\sigma_u^2 = \text{Var}(\varepsilon) + \left(\frac{\beta}{1+\psi}\right)^2 \text{Var}(\xi - w) - \frac{2\beta}{1+\psi}\text{Cov}(\varepsilon, \xi - w) = s_{11} + \widetilde{\beta}\left(\widetilde{\beta}s_{22}\widetilde{\kappa} - 2s_{12}\right) \quad \text{(A.13)}$$

To simplify this expression, we use the fact that

$$\sigma_{u\xi^*} \equiv \text{Cov}(u, \xi^*) = \text{Cov}\left(\varepsilon - \beta\left[\frac{\xi - w}{1+\psi}\right], \frac{\xi - w}{1+\psi}\right) = \left(\frac{1}{1+\psi}\right)(s_{12} - \widetilde{\beta}s_{22}\widetilde{\kappa}).$$

Rearranging, $\widetilde{\sigma}_{u\xi^*} \equiv (1+\psi)\sigma_{u\xi^*} = s_{12} - \widetilde{\beta}s_{22}\widetilde{\kappa}$. Substituting this into (A.13) along with $\widetilde{\beta} = (s_{13} - \sigma_{u\zeta})/s_{23}$ gives (A.2). Solving $\widetilde{\sigma}_{u\xi^*} = s_{12} - \widetilde{\beta}s_{22}\widetilde{\kappa}$ for $\widetilde{\beta}$ and equating this with $\widetilde{\beta} = (s_{13} - \sigma_{uz})/s_{23}$ gives (A.1). As explained in the proof of Proposition 2.1, (A.3) and (A.4) follow from (A.1) and (A.2) by substituting $\rho_{u\zeta} = \sigma_u\rho_{u\zeta}s_{33}$ and $\widetilde{\sigma}_{u\xi^*} = \rho_{u\xi^*}\sigma_u(\widetilde{\kappa}s_{22})^{1/2}$. The first of these equalities is simply the definition of $\rho_{u\zeta}$, so it suffices to verify the second. By our construction,

$$\text{Var}(\xi^*) = \text{Var}\left(\frac{\xi - w}{1+\psi}\right) = \left(\frac{1}{1+\psi}\right)^2 s_{22}\widetilde{\kappa} = \left(\frac{1}{1+\psi}\right)^2 s_{22}\left[(1+\psi)^2\kappa\right] = s_{22}\kappa$$

and hence

$$\widetilde{\sigma}_{u\xi^*} \equiv (1+\psi)\sigma_{u\xi^*} = \rho_{u\xi^*}\sigma_u(1+\psi)\sigma_{\xi^*} = \rho_{u\xi^*}\sigma_u\sqrt{s_{22}(1+\psi)^2\kappa} = \rho_{u\xi^*}\sigma_u\sqrt{s_{22}\widetilde{\kappa}}$$

as required. All that remains is to verify $\rho_{uv}^2 + \rho_{uz}^2 < 1$. To establish this, we show that our construction satisfies the expression for ρ_{uv} given Lemma A.1 (b). The required inequality then follows, given our choice of $\rho_{u\zeta}$ to satisfy (25), because the steps in the proof of Proposition 2.2 are reversible. By our construction of u and v from above,

$$\sigma_{uv} = \text{Cov}\left(\varepsilon - \beta\left[\frac{\xi - w}{1+\psi}\right], \left[\frac{\xi - w}{1+\psi} - \pi\zeta\right]\right) = \left(\frac{1}{1+\psi}\right) s_{12} - \widetilde{\beta}\left(\frac{1}{1+\psi}\right) s_{22}\widetilde{\kappa} - \pi s_{13} + \pi\widetilde{\beta} s_{23}.$$

Substituting our choices of π and $\widetilde{\beta}$ along with the expression for $\widetilde{\sigma}_{u\xi^*}$ used in our derivation of σ_u^2, this simplifies to

$$\sigma_{uv} = \left(\frac{1}{1+\psi}\right)\left(\widetilde{\sigma}_{u\xi^*} - \frac{s_{23}}{s_{33}}\sigma_{u\zeta}\right).$$

Substituting $\sigma_{uv} = \rho_{uv}\sigma_u\sigma_v$, $\widetilde{\sigma}_{u\xi^*} = \rho_{u\xi^*}\sigma_u(s_{22}\widetilde{\kappa})^{1/2}$, $\sigma_{u\zeta} = \rho_{u\zeta}\rho_u\sqrt{s_{33}}$ and re-arranging gives

$$\sigma_v\rho_{uv} = \left(\frac{1}{1+\psi}\right)\left[\rho_{u\xi^*}\left(s_{22}\widetilde{\kappa}\right)^{1/2} - \frac{s_{23}}{\sqrt{s_{33}}}\rho_{u\zeta}\right].$$

The desired result follows since $\sigma_v = \left[s_{22}(\widetilde{\kappa} - r_{23}^2)\right]^{1/2}/(1+\psi)$ as shown above. □

Proof of Corollary 2.1. This argument is a special case of the reasoning from the proof of Proposition B.1 with $\mathcal{R} = (L, 1] \times [-1, 1]$. We rely on one additional fact, namely that $g(L) = -\text{sign}\left\{r_{12}r_{23} - Lr_{13}\right\}$ which follows from some simple algebra. First suppose that $r_{12}r_{23} < Lr_{13}$. In this case g is positive for all $x_1 \in (L, 1]$. If x_2^* is interior, then x_1^* is L or 1. But in this case $g(L) = 1$ so the maximum must occur at (L, x_2^*). Having found the maximum, we now require the minimum. The minimum could equal $g(1)$. Alternatively it could occur at a corner solution for x_2^*, in which case f simplifies to $f(x_1, 1) = r_{23}/\sqrt{x_1}$ or $f(x_1, -1) = -r_{23}/\sqrt{x_1}$ depending on whether x_2 equals 1 or -1. One of these two functions is negative. In contrast, $g(1)$ is positive so it cannot be the minimum: by inspection the minimum occurs at $-|r_{23}|/\sqrt{L}$. Analogous reasoning holds in the case of $r_{12}r_{23} > Lr_{13}$. If $r_{21}r_{23} = Lr_{13}$, then $f(x_1, x_2) = x_2 r_{23}/\sqrt{x_1}$ so we can again find the extrema by inspection. □

Proof of Corollary 2.2. See the proof of Proposition B.2, of which Corollary 2.2 is a special case. □

Proof of Lemma 3.1. By the law of total probability,

$$\begin{aligned}\text{Cov}(T^*, T) = (1-\alpha_1)p^* - pp^* &= \left\{(1-\alpha_1) - \left[\alpha_0(1-p^*) + (1-\alpha_1)p^*\right]\right\} p^* \\ &= p^*(1-p^*)(1-\alpha_0-\alpha_1) = \text{Var}(T^*)(1-\alpha_0-\alpha_1)\end{aligned}$$

and therefore

$$\psi = \frac{\text{Cov}(T^*, \widetilde{w})}{\text{Var}(T^*)} = \frac{\text{Cov}(T^*, T)}{\text{Var}(T^*)} - 1 = \frac{\text{Var}(T^*)(1-\alpha_0-\alpha_1)}{\text{Var}(T^*)} - 1 = -(\alpha_0+\alpha_1)$$

by the definition of $\widetilde{w}$ from (3), establishing part (i). For part (ii), first note that $\widetilde{w}$ can only take on the values $\{-1, 0, 1\}$ yielding

$$\begin{aligned}\mathbb{E}[\widetilde{w}] &= \mathbb{P}(\widetilde{w} = 1) - \mathbb{P}(\widetilde{w} = -1) = \mathbb{P}(T = 1, T^* = 0) - \mathbb{P}(T = 0, T^* = 1) \\ &= \alpha_0(1 - p^*) - \alpha_1 p^* = \alpha_0 - (\alpha_0 + \alpha_1)p^*\end{aligned}$$

from which we obtain

$$\tau \equiv \mathbb{E}[\widetilde{w}] - \psi \mathbb{E}[T^*] = [\alpha_0 - (\alpha_0 + \alpha_1)p^*] + (\alpha_0 + \alpha_1)p^* = \alpha_0.$$

Finally,

$$w \equiv \widetilde{w} - \tau - \psi T^* = (T - T^*) - \alpha_0 + (\alpha_0 + \alpha_1)T^* = (T - \alpha_0) - (1 - \alpha_0 - \alpha_1)T^*$$

establishing (iii). □

Proof of Lemma 3.2. By the law of total probability, $p = \alpha_0(1 - p^*) + (1 - \alpha_1)p^*$. Rearranging this equality gives (i). For part (ii), first notice that $\sigma_w^2 = \mathbb{E}(w^2)$ since w is mean zero by construction. Now, using Lemma 3.1 (iii) the probability mass function of w is

$$\begin{aligned}\mathbb{P}(w = -\alpha_0) &= \mathbb{P}(T = 0, T^* = 0) = (1 - \alpha_0)(1 - p^*) \\ \mathbb{P}(w = \alpha_1 - 1) &= \mathbb{P}(T = 0, T^* = 1) = \alpha_1 p^* \\ \mathbb{P}(w = 1 - \alpha_0) &= \mathbb{P}(T = 1, T^* = 0) = \alpha_0(1 - p^*) \\ \mathbb{P}(w = \alpha_1) &= \mathbb{P}(T = 1, T^* = 1) = (1 - \alpha_1)p^*\end{aligned}$$

and accordingly we have

$$\begin{aligned}\mathbb{E}(w^2) &= \alpha_0^2(1 - \alpha_0)(1 - p^*) + (1 - \alpha_1)^2\alpha_1 p^* + (1 - \alpha_0)^2\alpha_0(1 - p^*) + \alpha_1^2(1 - \alpha_1)p^* \\ &= p^*\alpha_1(1 - \alpha_1) + (1 - p^*)\alpha_0(1 - \alpha_0)\end{aligned}$$

after expanding and simplifying. Eliminating p^* using part (i) gives

$$\sigma_w^2 = \frac{1}{1 - \alpha_0 - \alpha_1}\left[(p - \alpha_0)\alpha_1(1 - \alpha_1) + (1 - p - \alpha_1)\alpha_0(1 - \alpha_0)\right]$$

from which (ii) follows after straightforward but tedious algebra. □

Proof of Proposition 3.1. To begin we show that p^* cannot equal zero or one. By Assumption 2.1 (iv), ξ^* must be identically zero if $p^* \in \{0, 1\}$. But since $\xi^* = \pi\zeta + \upsilon$ by Equation 14, this can only occur if $|\text{Cor}(\zeta, \upsilon)| = 1$ which is ruled out by Assumption 2.3. Similarly, the positive definiteness of Σ implies that $p \notin \{0, 1\}$. Now, solving Lemma 3.2 (b) for α_0 and α_1 in turn, we obtain

$$\alpha_0 = \frac{\sigma_w^2 - p\alpha_1}{1 - p - \alpha_1}, \quad \alpha_1 = \frac{\sigma_w^2 - (1 - p)\alpha_0}{p - \alpha_0}.$$

where $\sigma_w^2 = s_{22}(1 - \widetilde{\kappa})$ by (20). By Lemma 3.2 (a) it follows that $\alpha_0 < p$ and $\alpha_1 < 1 - p$

since $0 < p^* < 1$, so neither denominator can be zero. Now, viewing α_1 as a function of α_0,

$$\frac{\partial}{\partial \alpha_0}\alpha_1 = \frac{\sigma_w^2 - p(1-p)}{(p-\alpha_0)^2}, \quad \frac{\partial^2}{\partial \alpha_0^2}\alpha_1 = 2\left[\frac{\sigma_w^2 - p(1-p)}{(p-\alpha_0)^3}\right]$$

so we see that the signs of the first and second derivatives are entirely determined by the sign of $\sigma_w^2 - p(1-p)$. Since $T = \tau + (1+\psi)T^* + w$ where $\text{Cov}(T^*, w) = 0$, it follows that

$$\text{Var}(T) = p(1-p) = (1+\psi)^2\text{Var}(T^*) + \text{Var}(w) = (1-\alpha_0-\alpha_1)^2 p^*(1-p^*) + \sigma_w^2$$

Since $p^* \notin \{0, 1\}$, we have $\sigma_w^2 - p(1-p) < 0$. Thus α_1 is a strictly decreasing and strictly concave function of α_0 on the interval $\alpha_0 \in [0, p)$. Evaluating this function at $\alpha_0 = 0$ we obtain $\alpha_1 = s_{22}(1-\widetilde{\kappa})/p$. Setting $\alpha_1 = 0$ and solving for α_0, we obtain $\alpha_0 = s_{22}(1-\widetilde{\kappa})/(1-p)$. These are the α_0 and α_1 axis intercepts, respectively, in Figure 1. Note that both are non-negative since $s_{22} \geq 0$ and $\widetilde{\kappa} \leq 1$. Since s_{22} is the variance of the residual from a projection of T on $\mathbf{x}$, we know that $s_{22} \leq p(1-p)$. And because $0 \leq L \leq 1$, it follows that $s_{22}(1-L)/(1-p) \leq p$ and similarly that $s_{22}(1-L)/p \leq (1-p)$. Hence,

$$0 \leq \alpha_0 < s_{22}(1-L)/(1-p) < p, \quad 0 \leq \alpha_1 < s_{22}(1-L)/p < 1-p.$$

These two values cannot occur simultaneously, however. For any value of σ_w^2 less than $s_{22}(1-L)$ the curve relating α_0 and α_1 necessarily lies below the function $E(\alpha_0) = [s_{22}(1-L) - (1-p)]\alpha_0/(p-\alpha_0)$, since

$$\alpha_1 = \frac{\sigma_w^2 - (1-p)\alpha_0}{p-\alpha_0} < \frac{s_{22}(1-L) - (1-p)\alpha_0}{p-\alpha_0}.$$

The function E is the outer envelope given by the dashed black curve in Figure 1, which cannot actually be attained since $\widetilde{\kappa} > L$ by Proposition 2.2. Fixing $\widetilde{\kappa}$ determines a functional relationship between α_0 and α_1. To find the corresponding bound for ψ we use the fact that $\psi = -(\alpha_0+\alpha_1)$ by Lemma 3.1 (i). Since α_1 is a strictly concave function of α_0, the minimum value of $\alpha_0 + \alpha_1$ is a corner solution: either $s_{22}(1-\widetilde{\kappa})/p$ or $s_{22}(1-\widetilde{\kappa})/(1-p)$ depending on whether p is larger than $1-p$. Again because the function is strictly concave, the maximum value of $\alpha_0 + \alpha_1$ could be either interior or occur at the *opposite* corner. To solve for an interior maximum, we substitute the constraint $\alpha_1 = [s_{22}(1-\widetilde{\kappa}) - (1-p)\alpha_0]/(p-\alpha_0)$ into the objective function to yield

$$(\alpha_0 + \alpha_1) = \alpha_0 + [s_{22}(1-\widetilde{\kappa}) - (1-p)\alpha_0]/(p-\alpha_0)$$

Differentiating the right hand side with respect to α_0 gives the first order condition

$$(p-\alpha_0)^2 + s_{22}(1-\widetilde{\kappa}) - p(1-p) = 0.$$

which is a quadratic in α_0 with roots $\alpha_0 = p \pm \sqrt{p(1-p) - s_{22}(1-\widetilde{\kappa})}$. Since $p(1-p) > \sigma_w^2$ both of these are real. However, the "+" root violates the constraint $\alpha_0 < p$, hence the unique solution is the "−" root. Substituting this into the constraint, we obtain the corresponding solution for α_1. Hence, an interior maximum of $(\alpha_0 + \alpha_1)$ occurs at

$$\alpha_0 = p - \sqrt{p(1-p) - s_{22}(1-\widetilde{\kappa})}, \quad \alpha_1 = (1-p) - \sqrt{p(1-p) - s_{22}(1-\widetilde{\kappa})}$$

Note that $\alpha_0 \geq 0$ iff $s_{22}(1-\widetilde{\kappa}) > p(1-2p)$. Similarly, $\alpha_1 \geq$ iff $s_{22}(1-\widetilde{\kappa}) > (1-p)(2p-1)$. Hence, the maximum value of $(\alpha_0 + \alpha_1)$ is interior iff $s_{22}(1-\widetilde{\kappa}) > m(p)$, in which case $(\alpha_0 + \alpha_1) = 2\sqrt{p(1-p)s_{22}(1-\widetilde{\kappa})} - 1$. □

B Additional Results

This appendix contains proofs of two additional results that do not appear in our paper but are used in the associated R package `ivdoctr`: Proposition B.1 and Proposition B.2. These propositions provide bounds for $\rho_{u\zeta}$ and β under interval restrictions on $\widetilde{\kappa}$ and $\rho_{u\xi^*}$.

Proposition B.1. *Suppose that $(\widetilde{\kappa}, \rho_{u\xi^*})$ is known,* a priori, *to lie in a set $\mathcal{R}$ that takes the form $\mathcal{R} \equiv [\widetilde{a}, \widetilde{b}] \times [c^*, d^*] \subseteq (L, 1] \times [-1, 1]$. Then, under the conditions of Theorem 2.1, the sharp identified set for $\rho_{u\zeta}$ is the open interval $(\min_S f, \max_S f)$ where*

$$f(\widetilde{\kappa}, \rho_{u\xi^*}) \equiv \frac{r_{23}\rho_{u\xi^*}}{\widetilde{\kappa}^{1/2}} - (r_{12}r_{23} - r_{13}\widetilde{\kappa}) \left[\frac{1-\rho_{u\xi^*}^2}{\widetilde{\kappa}\,(\widetilde{\kappa} - r_{12}^2)}\right]^{1/2}$$

and S is a finite set defined by $S = S_1 \cup S_2 \cup \left\{ \{\widetilde{a}, \widetilde{b}\} \times \{c^*, d^*\} \right\}$ where S_1 is given by

$$S_1 \equiv \mathcal{R} \cap \left\{ \left(\widetilde{a}, h(\widetilde{a})\right), \left(\widetilde{b}, h(\widetilde{b})\right) \right\}, \quad h(\widetilde{\kappa}) \equiv \frac{-r_{23}\left(\widetilde{\kappa} - r_{12}^2\right)^{1/2}}{\left[(r_{12}r_{23} - \widetilde{\kappa}r_{13})^2 + r_{23}^2(\widetilde{\kappa} - r_{12}^2)\right]^{1/2}}$$

and S_2 is given by

$$S_2 \equiv \mathcal{R} \cap \left(\{\Xi(c^*) \times \{c^*\}\} \cup \{\Xi(d^*) \times \{d^*\}\}\right)$$

where $\Xi(c^*)$ and $\Xi(d^*)$ denote the set of roots of

$$(1-\rho_{u\xi^*}^2)\left[(2r_{12}r_{23} - r_{13}r_{12}^2)\widetilde{\kappa} - r_{23}r_{12}^3\right]^2 - \rho_{u\xi^*}^2 r_{23}(\widetilde{\kappa} - r_{12})^3 = 0$$

with $\rho_{u\xi^*}$ held fixed at c^* and d^*, respectively.

Proof of Proposition B.1. To simplify the notation in this argument, we adopt the shorthand $x_1 \equiv \widetilde{\kappa}$ and $x_2 \equiv \rho_{u\xi^*}$ and accordingly write $f(x_1, x_2)$ in place of $f(\widetilde{\kappa}, \rho_{u\xi^*})$. Similarly, we write $[a, b]$ and $[c, d]$ in place of $[\widetilde{a}, \widetilde{b}]$ and $[c^*, d^*]$. Let (x_1^*, x_2^*) be an extremum of f and define $\widehat{x}_1 = r_{12}r_{23}/r_{13}$. There are two possibilities: either x_2^* is interior or it lies on the boundary. We begin by showing that if x_2^* is interior, x_1^* must lie on the boundary.

If x_2^* is interior, then it must satisfy the first order condition

$$\frac{\partial f(x_1, x_2)}{\partial x_2} = \frac{r_{23}}{\sqrt{x_1}} + \left[\frac{(r_{12}r_{23} - r_{13}x_1)}{\sqrt{x_1(x_1 - r_{12}^2)}}\right]\left[\frac{x_2}{\sqrt{1-(x_2)^2}}\right] = 0.$$

We can assume $x_1^* \neq \widehat{x}_1$, since $x_1^* = \widehat{x}_1$ implies $f(x_1, x_2) = r_{23}x_2/\widehat{x}_1^{1/2}$, in which case x_2^* could not be interior. Solving the first-order condition, we obtain $x_2^* = h(x_1^*)$ where

$$h(x_1) = \frac{-r_{23}\left(x_1 - r_{12}^2\right)^{1/2}}{\left[(r_{12}r_{23} - r_{13}x_1)^2 + r_{23}^2(x_1 - r_{12}^2)\right]^{1/2}},$$

eliminating an extraneous solution by noting that x_2^* must have the same sign as the ratio $-r_{23}/(r_{12}r_{23} - r_{13}x_1^*)$. Concentrating x_2 out of f, we have

$$g(x_1) \equiv f\big(x_1, h(x_1)\big) = -\text{sign}\{r_{12}r_{23} - x_1 r_{13}\}\sqrt{\frac{(r_{12}r_{23} - x_1r_{13})^2 + r_{23}(x_1 - r_{12}^2)}{x_1(x_1 - r_{12}^2)}}.$$

Differentiating and simplifying gives

$$g'(x_1) = -\frac{(L - r_{12}^2)(1 - r_{13}^2)}{2g(x_1)(x_1 - r_{12}^2)}$$

There are three cases. If $b < \widehat{x}_1$, then g' is strictly positive on $[a, b]$ and hence g is monotonically increasing on this interval, implying that x_1^* must lie on the boundary. If instead $\widehat{x}_1 < a$, then g' is strictly negative on $[a, b]$ and hence g is monotonically deacreasing on this interval, likewise implying that x_1^* must lie on the boundary. The remaining case is $a \leq \widehat{x}_1 \leq b$. Note that g is strictly increasing for $x_1 \in [a, \widehat{x}_1)$ and strictly decreasing for $x_1 \in (\widehat{x}_1, b]$. In this case we obtain candidate minima at $x_1 = a$ and $x_1 = b$ but not candidate maxima. This completes our characteriztion of candidate extrema for interior x_2^*.

Now suppose that x_2^* occurs at a corner. One possibility is that x_1^* likewise occurs at a corner; the other is that x_1^* is interior. In the latter case, it must satisfy the first order condition

$$\frac{\partial f(x_1, x_2)}{\partial x_1} = \frac{-r_{23}x_2}{2x_1^{2/3}} + \left\{\frac{r_{13}}{\sqrt{x_1(x_1 - r_{12}^2)}} + \frac{(r_{12}r_{23} - x_1r_{13})(2x_1 - r_{12}^2)}{2[x_1(x_1 - r_{12}^2)]^{3/2}}\right\}\sqrt{1 - x_2^2}$$

and hence the roots of the polynomial

$$(1 - x_2^2)[(2r_{12}r_{23} - r_{13}r_{12}^2)x_1 - r_{23}r_{12}^3]^2 - x_2^2 r_{23}^2(x_1 - r_{12}^2)^3 = 0$$

holding x_2 fixed at c and d are likewise candidate extrema. Finally, since f is a continuous function, any value of $\rho_{u\zeta}$ within the resulting bounds can be attained. □

Proposition B.2. *Suppose that* $(\widetilde{\kappa}, \rho_{u\xi^*})$ *is known,* a priori, *to lie in a set* $\mathcal{R}$ *that takes the form* $\mathcal{R} \equiv [\widetilde{a}, \widetilde{b}] \times [c^*, d^*] \subseteq (L, 1] \times [-1, 1]$. *Then, under the conditions of Theorem 2.1, the sharp identified set for* $\widetilde{\beta}$ *is given by*

$$\mathcal{B} = \begin{cases} (-\infty, \infty), & \text{if } [c^*, d^*] = [-1, 1] \\ \left[\dfrac{s_{13}}{s_{23}} - \max\limits_{Q} g,\ \dfrac{s_{13}}{s_{23}} - \min\limits_{Q} g\right], & \text{otherwise} \end{cases}$$

where

$$g(\widetilde{\kappa}, \rho_{u\xi^*}) \equiv \frac{\sqrt{s_{11}s_{33}}}{\widetilde{\kappa}s_{23}} \left[r_{23}\sqrt{\widetilde{\kappa} - r_{12}^2} \left(\frac{\rho_{u\xi^*}}{\sqrt{1-\rho_{u\xi^*}^2}} \right) - (r_{12}r_{23} - \widetilde{\kappa}r_{13}) \right]$$

and Q is a finite set defined by $Q = Q_1 \cup \left\{ \{\widetilde{a}, \widetilde{b}\} \times \{c^*, d^*\} \right\}$ where Q_1 is given by

$$Q_1 \equiv \mathcal{R} \cap (\{\Psi(c^*) \times \{c^*\}\} \cup \{\Psi(d^*) \times \{d^*\}\})$$

with

$$\Psi(\rho_{u\xi^*}) = \left\{ 2r_{12}^2 \left(1 - \sqrt{1-\rho_{u\xi^*}^2}\right) / \rho_{u\xi^*}^2,\; 2r_{12}^2 \left(1 + \sqrt{1-\rho_{u\xi^*}^2}\right) / \rho_{u\xi^*}^2 \right\}.$$

Proof of Proposition B.2. To simplify the notation in this argument, we adopt the shorthand $x_1 \equiv \widetilde{\kappa}$ and $x_2 \equiv \rho_{u\xi^*}$ and accordingly write $g(x_1, x_2)$ in place of $g(\widetilde{\kappa}, \rho_{u\xi^*})$. Similarly, we write $[a, b]$ and $[c, d]$ in place of $[\widetilde{a}, \widetilde{b}]$ and $[c^*, d^*]$.

Begin by noticing that for any fixed x_1, g is a strictly monotonic function of x_2. This means that the extrema of g lie on the boundary for x_2. Suppose first that $[c, d] = [-1, 1]$. If $r_{23} > 0$, g is strictly increasing in x_2 and for any $x_1 \in (L, 1]$ we have $\lim_{x_2 \to -1} = -\infty$ and $\lim_{x_2 \to 1} = +\infty$. If $r_{23} < 0$, then g is strictly decreasing and the limits are reversed. Hence $\mathcal{B} = (-\infty, \infty)$. Now suppose that $[c, d]$ is a strict subset of $(-1, 1)$. In this case we characterize the optimal values of x_1 at $x_2 = c$ and $x_2 = d$. Since $g(x_1, 0) = \sqrt{s_{11}}\,(r_{13} - r_{12}r_{23}/x_1)$, the extrema of g as a function of x_1 when $x_2 = 0$ occur at a and b. If instead $x_2 \neq 0$, the extrema could still occur at a and b, or they could be interior. If interior, they must satisfy the first order condition

$$x_1^2/4 - r_{12}^2 x_1/x_2^2 + r_{12}^4/x_2^2 = 0$$

yielding the set of solutions

$$\Psi(x_2) = \left\{ 2r_{12}^2 \left(1 - \sqrt{1-x_2^2}\right) / x_2^2,\; 2r_{12}^2 \left(1 + \sqrt{1-x_2^2}\right) / x_2^2 \right\}.$$

Hence, it suffices to evaluate g at all elements of $\mathcal{R} \cap (\{\Psi(c) \times \{c\}\} \cup \{\Psi(d) \times \{d\}\})$ and at the corners $\{a, b\} \times \{c, d\}$. Since g is a continuous function, any point within the bounds for β can be attained. □

Proposition B.3. *Suppose that $(\widetilde{\kappa}, \rho_{u\xi^*})$ is known,* a priori, *to lie in $\mathcal{R} \equiv [\widetilde{a}, \widetilde{b}] \times [c^*, d^*] \subseteq (L, 1] \times [-1, 1]$. Then, under the conditions of Proposition 3.1,*

$$\min_{[\widetilde{a}, \widetilde{b}]} \underline{\beta}(\widetilde{\kappa}) \leq \beta \leq \max_{[\widetilde{a}, \widetilde{b}]} \overline{\beta}(\widetilde{\kappa})$$

where $\underline{\beta}(\widetilde{\kappa}) \equiv \min B(\widetilde{\kappa})$, $\overline{\beta}(\widetilde{\kappa}) \equiv \max B(\widetilde{\kappa})$,

$$\begin{aligned} B(\widetilde{\kappa}) &= \left\{ (1+\psi)(s_{13}/s_{23} - g) \colon \psi \in \left\{ \underline{\psi}(\widetilde{\kappa}), \overline{\psi}(\widetilde{\kappa}) \right\},\; g \in \left\{ \underline{g}(\widetilde{\kappa}), \overline{g}(\widetilde{\kappa}) \right\} \right\} \\ \underline{g}(\widetilde{\kappa}) &= \min\{g(\widetilde{\kappa}, c^*),\; g(\widetilde{\kappa}, d^*)\} \\ \overline{g}(\widetilde{\kappa}) &= \max\{g(\widetilde{\kappa}, c^*), g(\widetilde{\kappa}, d^*)\} \end{aligned}$$

and g is as defined in Proposition B.2

Proof of Proposition B.3. This follows from Proposition 3.1 along with the fact that g is monotonic in $\rho_{u\xi^*}$ for fixed $\widetilde{\kappa}$ and $\beta = (1+\psi)[s_{13}/s_{23} - g(\widetilde{\kappa}, \rho_{u\xi^*})]$. □

C Uniform Draws on the Conditional Identified Set

In this appendix we provide details of our method for making uniform draws on $\Theta(\boldsymbol{\varphi}^{(j)})$, an ingredient of our procedure for carrying out inference for $\boldsymbol{\theta}$ from subsection 4.2. We first describe the classical measurement error case and then explain what changes in the case of a binary T^*. In the classical measurement error case, $\psi = 0$ so that $\widetilde{\kappa} = \kappa$. Thus, equation (25) describes a manifold relating $\rho_{u\zeta}, \rho_{u\xi^*}$ and κ. To draw uniformly on this manifold, subject to researcher beliefs, we proceed as follows. Let $\mathcal{R}$ denote a rectangular region encoding interval restrictions on κ and $\rho_{u\xi^*}$. We first draw uniformly on $\mathcal{R}$, and then re-weight these draws based on the local surface area of the manifold at each draw $(\rho_{u\xi^*}^{(\ell)}, \kappa^{(\ell)})$. By local surface area we refer to

$$M\left(\rho_{u\xi^*}, \kappa\right) = \sqrt{1 + \left(\frac{\partial \rho_{u\xi}}{\partial \rho_{u\xi^*}}\right)^2 + \left(\frac{\partial \rho_{u\zeta}}{\partial \kappa}\right)^2}. \tag{C.1}$$

The derivatives required to evaluate the function M are

$$\frac{\partial \rho_{u\zeta}}{\partial \rho_{u\xi^*}} = \frac{\rho_{Tz}}{\sqrt{\kappa}} + \frac{\rho_{u\xi^*}\left(r_{12}r_{23} - \kappa r_{13}\right)}{\sqrt{\kappa\left(\kappa - r_{12}^2\right)\left(1 - \rho_{u\xi^*}^2\right)}}$$

$$\frac{\partial \rho_{u\zeta}}{\partial \kappa} = -\frac{\rho_{u\xi^*} r_{23}}{2\kappa^{3/2}} + \sqrt{\frac{1 - \rho_{u\xi^*}^2}{\kappa\left(\kappa - r_{12}^2\right)}}\left\{r_{13} + \frac{1}{2}\left(r_{12}r_{23} - \kappa r_{13}\right)\left[\frac{1}{\kappa} + \frac{1}{\kappa - r_{12}^2}\right]\right\}.$$

To accomplish the re-weighting, we first evaluate $M^{(\ell)} = M(\rho_{u\xi^*}^{(\ell)}, \kappa^{(\ell)})$ at each draw ℓ that was accepted in the first step. We then calculate $M_{max} = \max_{\ell=1,\ldots,L} M^{(\ell)}$ and *resample* the draws $\left(\rho_{u\zeta}^{(\ell)}, \rho_{u\xi^*}^{(\ell)}, \kappa^{(\ell)}\right)$ with probability $p^{(\ell)} = M^{(\ell)}/M_{max}$. Now suppose that T^* is binary, so that the measurement error is not classical. In this case we proceed in two steps. First, we generate draws on the manifold relating $(\rho_{u\xi^*}, \rho_{u\zeta}, \widetilde{\kappa})$ *exactly* as in the classical measurement error case, by simply replacing κ with $\widetilde{\kappa}$ in the preceding equations. Given a draw $(\rho_{u\zeta}^{(\ell)}, \rho_{u\xi^*}^{(\ell)}, \widetilde{\kappa}^{(\ell)})$ we then generate the corresponding $\psi^{(\ell)}$ by drawing uniformly on the interval $\left[\underline{\psi}(\widetilde{\kappa}^{(\ell)}),\ \overline{\psi}(\widetilde{\kappa}^{(\ell)})\right]$ defined in Proposition 3.1.

D Bayesian versus Frequentist Inference

Under certain assumptions our inferences for the identified set from subsection 4.1 can be given a Frequentist repeated-sampling interpretation in the limit under the posterior for Σ described in subsection 4.4. We now give a brief overview of how this can be achieved, appealing to results from Kline and Tamer (2016). Alternatively, one could follow the closely related approach of Kitagawa (2012).

Let $\boldsymbol{\varphi}_0$ denote the "true" value of the reduced form parameter vector, i.e. the solution to the population maximum likelihood criterion function. In our example, this corresponds to the true reduced form covariance matrix Σ. Under weak regularity conditions on the true data generating process for $(y, T, \mathbf{x}, z)$, our inverse-Wishart posterior is consistent for $\boldsymbol{\varphi}_0$ by Doob's Theorem.[21] Now let $\widehat{\boldsymbol{\varphi}}_n$ denote the maximum likelihood estimator based on a sample of n observations. In our example this corresponds to the sample covariance matrix $S/(n-k)$ of the regression residuals $Y - X\widehat{B}$. Because our prior is continuous with full support and our posterior is consistent for $\boldsymbol{\varphi}_0$, Hartigan (1983) Theorem 11.2 establishes that $\sqrt{n}(\boldsymbol{\varphi} - \widehat{\boldsymbol{\varphi}}_n)$ is asymptotically normal under weak regularity conditions on the true data generating process. Crucially, this holds *regardless* of whether the likelihood is correctly specified: the required regularity conditions are effectively identical to those used to establish the asymptotic normality of the Frequentist quasi-maximum likelihood estimator. Hence, under mild conditions both the Bayesian posterior and Frequentist maximum likelihood estimator are asymptotically normal. Now, let J denote the information matrix, and let H denote the expected Hessian. When the information matrix equality $H = -J$ holds, the Bayesian posterior and Frequentist large-sample distributions agree: both have variance matrix J^{-1}. In this case, we appeal to Theorem 5 of Kline and Tamer (2016) to show that a $(1-\delta)$ credible set for Θ is also an exact pointwise $(1-\delta)$ Frequentist confidence set.[22]

If the normal likelihood for the reduced form errors is correctly specified, then the information matrix equality holds. Correct specification, however, is not a necessary condition. Let $\widehat{s}_{ij}$ and $\widehat{s}_{lm}$ be the maximum likelihood estimators of two arbitrary elements s_{ij} and s_{jm} of the reduced form covariance matrix Σ. The necessary and sufficient condition for Bayesian posterior and Frequentist inference for $\boldsymbol{\varphi}$ to agree in our example is that the asymptotic covariance between $\widehat{s}_{ij}$ and $\widehat{s}_{lm}$ equals $(s_{ij}s_{jm} + s_{im}s_{jl})$. When this condition fails, the equivalence between credible sets and confidence intervals described in the preceding paragraph no longer holds. A solution to this problem is to avoid explicitly specifying a prior and likelihood and instead sample $\boldsymbol{\varphi}^{(j)}$ from a multivariate normal distribution constructed to exactly match the Frequentist asymptotic distribution. This idea corresponds to the "pragmatic Bayesian" approach described by Sims (2010) and the "artificial 'sandwich' posterior" of Müller (2013). While we are in general supportive of this idea, we do not adopt it here for two reasons. First, implementing it in our examples would require us to rely on estimated fourth-order moments of the distribution of $(\varepsilon, \xi, \zeta)$, which are likely to be unreliable in practice. Second, our partial identification bounds rely crucially on the positive definiteness

[21]See Hartigan (1983) 4.4 for regularity conditions sufficient for Doob's Theorem.

[22]Formally, one must first verify an asymptotic independence property given in Assumption 5 of Kline and Tamer (2016). The examples considered in the present paper, however, fall under the case discussed in Remark 5 and Lemma 1 from Kline and Tamer (2016), so that one only requires the validity of both the usual Frequentist delta-method, and its Bayesian analogue.

of Σ, but drawing the half-vectorization of this matrix, $\text{vech}(\Sigma)$, from a multivariate normal distribution can produce draws that violate this restriction.

As in the classical measurement error case, we draw the reduced form covariance matrix from an Inverse-Wishart posterior when T^* is binary. Of course, the distribution of U cannot in fact be normal if any of the variables (y, T, z) is discrete. Nevertheless, the posterior for the reduced form parameters will still be asymptotically normal, centered at the maximum likelihood estimates. Provided that the aforementioned condition on the asymptotic covariance between $\widehat{s}_{ij}$ and $\widehat{s}_{lm}$ holds *approximately*, this asymptotically normal posterior will likewise approximate the Frequentist large-sample distribution. One could, in principle, write down a different likelihood for the binary T^* case. But this would require one to model the distribution of $T^*|\mathbf{x}$, an object over which applied researchers are typically agnostic when reporting OLS and IV results. For this reason, we prefer to treat the continuous and binary T^* cases within a common framework. Note, however, that the bounds for ψ from Proposition 3.1 involve p. We suggest adopting an empirical Bayes approach and setting p equal to the sample analogue $\widehat{p}$. This is irrelevant from a large-sample perspective, and amounts to a rounding error in applications. When the exogenous covariates $\mathbf{x}$ include only a constant, p equals φ_T, so one could obtain posterior draws for this parameter directly from our normal-Jeffreys model. In the general case, however, it is less straightforward to obtain posterior draws for p. For one, the reduced form regression for T is not a generative model: it could imply conditional probabilities that are outside of $[0, 1]$. Addressing this difficulty would require one to either adopt a non-parametric approach or to impose parametric assumptions on the distribution of $T|\mathbf{x}$. Moreover, converting the conditional probability $\mathbb{P}(T|\mathbf{x})$ into the unconditional probability p requires integrating over the distribution of $\mathbf{x}$. The additional complications required to incorporate posterior uncertainty over p for the general $\mathbf{x}$ seem excessive, particularly given that sampling uncertainty in p is of a smaller order than sampling uncertainty in Σ.

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